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2000 J. Phys. A: Math. Gen. 33 3045

(http://iopscience.iop.org/0305-4470/33/15/311)

J. Phys. A: Math. Gen. **33** (2000) 3045–3052. Printed in the UK PII: S0305-4470(00)11710-X

Integrable systems without the Painlevé property

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Received 9 February 2000

Abstract. We examine whether the Painlevé property is a necessary condition for the integrability of nonlinear ordinary differential equations. We show that for a large class of linearizable systems this is not the case. In the discrete domain, we investigate whether the singularity confinement property is satisfied for the discrete analogues of the non-Painlevé continuous linearizable systems. We find that while these discrete systems are themselves linearizable, they possess non-confined singularities.

1. Introduction

For over a century, the Painlevé property [1] has been the cornerstone of integrability. The reason Painlevé introduced this property, which later was named after him, was a question that was open at the time, and of particular interest: 'Is it possible to define (new) functions from the solutions of nonlinear differential equations?'

In some sense, this amounted to introducing the analogue of special functions into the nonlinear domain. The study of linear equations had shown where the difficulties lay [2]. In particular, one had to deal with the multivaluedness that could appear as a consequence of the singularities of the coefficients of the equation which, for linear equations, are the only possible singularities of the solutions. The extension of these ideas to the nonlinear domain appeared hopeless since the location of bad singularities could now depend on the initial conditions. Then Painlevé made a leap of faith by requesting that all critical (i.e. multivalued) movable (i.e. initial-condition-dependent) singularities be absent.

The ordinary differential equations (ODEs) without critical movable singularities are said to possess the Painlevé property. Their solutions define functions which in some cases (the Painlevé transcendents) cannot be expressed in terms of known functions. The precise way to integrate (i.e. to construct the solutions of) the ODEs with the Painlevé property can be very complicated [3] but the important fact is that this can in principle be done. Thus, the property came to be synonymous with integrability. At this point it must be made clear that the integrability we are talking about, related to the Painlevé property, is of a special kind often referred to as 'algebraic integrability' [4]. It is, for instance, the kind of integrability that characterizes systems integrable in terms of inverse scattering transform (IST) techniques [5]. However, in common practice, many other 'brands' of integrability do exist [6]. Integrability through quadratures, like that encountered in the case of finite-dimensional Hamiltonian systems, is of (relatively) frequent occurrence, and is not identical to algebraic integrability.

0305-4470/00/153045+08$30.00

Linearizability, i.e. the reduction of the system to a system of linear equations through a local transformation, is a further, different, type.

In this paper, we shall examine the relation of these kinds of integrability to the Painlevé property, focusing on linearizable systems. In the second part of the paper, we shall examine the discrete analogues of these notions. In this case, the role of the Painlevé property is played by singularity confinement [7]. The latter is believed to be a necessary condition for integrability (but unlike the Painlevé property it has turned out not to be sufficient as well [8]). We shall show that in both continuous and discrete settings, systems integrable through linearization can exist without the Painlevé property.

2. Integrable continuous systems and the Painlevé property

A first instance of integrability without the Painlevé property was the derivation of the integrable system described by the Hamiltonian [9]:

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + y^5 + y^3x^2 + \tfrac{3}{16}yx^4 \tag{2.1}$$

which has the second (besides the energy) constant of motion

$$C = -yp_x^2 + xp_xp_y + \tfrac{1}{2}y^4x^2 + \tfrac{3}{8}y^2x^4 + \tfrac{1}{32}x^6. \tag{2.2}$$

There are movable singularities where near some singular point t_0, one has $y \approx \alpha(t-t_0)^{-2/3}$, $x \approx \beta(t-t_0)^{-1/3}$ with $\alpha^3 = -\frac{2}{9}$, β arbitrary. Taking the cube of the variables is not sufficient to regularize them, however. Indeed, a detailed analysis of complex-time singularities shows that their expansions contain *all* powers of $(t-t_0)^{1/3}$. The fact that some multivaluedness was compatible with integrability led to the introduction of the notion of a 'weak Painlevé' property. However, it was soon realized [10] that (2.1) was a member of a vaster family of integrable Hamiltonian systems associated with the potential $V = (F(\rho+y) + G(\rho-y))/\rho$ where $\rho = \sqrt{x^2+y^2}$. Since the two functions F and G are free, one can easily show that the singularities of the solutions of the equations of motion can be arbitrary. The Hamiltonians of this family are integrable through quadratures and, in fact, the associated Hamilton–Jacobi equations are separable. This leads to the conclusion that this type of integrability is not necessarily related to the Painlevé property. (As a matter of fact, the same conclusion could have been reached if we had simply considered one-dimensional Hamiltonian systems.) One may justifiably argue that in the case of Hamiltonian systems the term integrability is to be understood as Liouville integrability which is not the one we refer to in relation to the Painlevé property. Still, Liouville integrability, and the dynamical symmetries to which it is associated, may be of utmost importance for physical applications and a systematic method for its detection would have been most welcome.

We turn now to a second case of integrability where the necessary character of the Painlevé property can be examined critically: that of linearizable systems. The term linearizable is used here to denote systems that can be reduced to linear equations through a local variable transformation. The first family of such systems are the projective ones [11]. Starting from the linear system for $(N+1)$ variables:

$$X'_\mu = \sum_{\nu=0}^{N} A_{\mu\nu}X_\nu \qquad \mu = 0, 1, \ldots, N \tag{2.3}$$

and introducing the quantities $x_\mu = X_\mu/X_0$ we obtain the projective Riccati system:

$$x'_\mu = a_\mu + \sum_{\nu=1}^{N} b_{\mu\nu}x_\nu + x_\mu \sum_{\nu=1}^{N} c_{\mu\nu}x_\nu \qquad \mu = 1, \ldots, N \tag{2.4}$$

where a_μ, $b_{\mu\nu}$ and $c_{\mu\nu}$ are given in terms of $A_{\mu\nu}$. As we have shown in [12] this system can be rewritten as a single Nth-order differential equation. For $N = 1$, this is just the Riccati equation for x_1:

$$x_1' = a_1 + b_{11}x_1 + c_{11}x_1^2. \tag{2.5}$$

For $N = 2$, the system can be reduced to equation VI of the Painlevé/Gambier classification [2]

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = -3w\frac{\mathrm{d}w}{\mathrm{d}z} - w^3 + q(z)\left(\frac{\mathrm{d}w}{\mathrm{d}z} + w^2\right) \tag{2.6}$$

for z some function of the independent variable of (2.4) and w a homographic function of x_1 with some specific functions of z as coefficients. Because of the underlying linearization, the projective Riccati systems possess the Painlevé property by construction. Indeed, X_μ have no movable singularities at all, and the only movable singularities of the x_μ are poles coming from the zeros of X_0.

However, there exists another kind of linearizability for which the Painlevé property need not be satisfied. Let us discuss the best-known second-order case. One of the equations of the Painlevé/Gambier classification, bearing the number XXVII, is the equation proposed by Gambier [13]:

$$x'' = \frac{n-1}{n}\frac{x'^2}{x} + \left(fx + \phi - \frac{n-2}{nx}\right)x' - \frac{nf^2}{(n+2)^2}x^3 + \frac{n(f' - f\phi)}{n+2}x^2 + \psi x - \phi - \frac{1}{nx} \tag{2.7}$$

where f, ϕ and ψ are definite rational functions of two arbitrary analytic functions and of their derivatives [2]. As Gambier has shown, equation (2.7) can be written as a system of two Riccati equations in cascade:

$$y' = -y^2 + \phi y + \frac{2f}{n(n+2)} + \frac{\psi}{n} \tag{2.8a}$$

$$x' = \frac{nf}{n+2}x^2 + nyx + 1. \tag{2.8b}$$

Gambier has shown that unless the parameter n appearing in (2.7) and (2.8) is integer, the equation does not possess the Painlevé property. (We must point out that this is a first necessary condition and, in general, not a sufficient one: further constraints on the coefficients are needed in order to ensure the Painlevé property.) On the other hand, the integration of the two Riccati equations in cascade can always be performed, through reduction to linear second-order equations. Thus, although the solution of (2.8) does not in general lead to a well defined function as a solution of (2.7), it can still be obtained in cascade.

Once the Painlevé property is deemed unnecessary for the linearization of the Gambier system, it is straightforward to extend the latter to the form

$$y' = \alpha y^2 + \beta y + \gamma \tag{2.9a}$$

$$x' = a(y,t)x^2 + b(y,t)x + c(y,t) \tag{2.9b}$$

where α, β and γ are arbitrary functions of t, while a, b and c are arbitrary functions of y and t. The integration in cascade of (2.9) can be obtained as previously. As a matter of fact, an extension like (2.9) gives the handle to the $(N{+}1)$-variables generalization of the Gambier system:

$$x_0' = a_0(t)x_0^2 + b_0(t)x_0 + c_0(t)$$
$$x_\mu' = a_\mu(x_0, \ldots, x_{\mu-1}, t)x_\mu^2 + b_\mu(x_0, \ldots, x_{\mu-1}, t)x_\mu + c_\mu(x_0, \ldots, x_{\mu-1}, t) \qquad \mu = 1, \ldots, N \tag{2.10}$$

where a_μ, b_μ and c_μ are arbitrary functions of their arguments. Again, the system (2.10) does not possess, generically, the Painlevé property while it can be linearized and integrated in cascade.

Until now, we have presented rather straightforward generalizations of integrable systems which violate the Painlevé property while preserving their linearizability. We shall close this section by introducing a new (at least to our knowledge) method of linearization which again leads to integrable systems not possessing the Painlevé property. Let us describe our general approach. The idea is the following: we start from a linear second-order equation in the form

$$\frac{\alpha x'' + \beta x' + \gamma x + \delta}{\epsilon x'' + \zeta x' + \eta x + \theta} = K \tag{2.11}$$

where $\alpha, \beta, \ldots, \theta$ are functions of t with K a constant, and a *nonlinear* second-order equation of the form

$$f(x'', x', x) = M \tag{2.12}$$

where f is a (possibly inhomogeneous) polynomial of degree two in x together with its derivatives, but linear in x'', and with M a constant. We then ask that the derivatives of both equations with respect to the independent variable, i.e. the resulting third-order equations, be identical up to an overall factor. This is a novel linearization approach. The explicit integration procedure is the following. We start from equation (2.12) with *given* M and initial conditions x_0, x_0' for some value t_0 of the independent variable t. We use (2.12) to compute x_0'' at t_0. Having these values, we can use (2.11) to compute the value of K. Since the latter is assumed to be a constant, we can integrate the linear equation (2.11) for all values of t. Since this solution will satisfy the third-order equation mentioned above, it will also be a solution of (2.12).

In order to illustrate this approach, we derive one equation that can be integrated through this linearization. Our starting assumption is that (2.12) contains a term $x''x'$. The more general term $x''(x' + cx + d)$ can always be reduced to this form, i.e. $c = d = 0$ through a rescaling and translation of x. It is then straightforward to obtain the full expression in the homogeneous subcase $\delta = \theta = 0$. We thus find

$$\frac{tx'' + (at - 1/2)x' + btx}{x'' + ax' + bx} = K \tag{2.13}$$

for the linear equation, and

$$x''x' + 2ax'^2 + 3bx'x + (2ab - b')x^2 = M \tag{2.14}$$

for the nonlinear one, with $b = a^2 - a'/2$ and a satisfying the equation

$$a''' = 6a''a + 7a'^2 - 16a'a^2 + 4a^4 \tag{2.15}$$

which is equation XII in the Chazy classification [14]. Its general solution can be obtained by putting $a = -u'/2u$. Equation (2.15) reduces to

$$u^{(\mathrm{IV})}u - u'''u' + u''^2/2 = 0 \tag{2.16}$$

which implies $u^{(V)} = 0$, so u is a quartic polynomial in the independent variable t with one constraint on its coefficients, because of (2.16). Given a and the corresponding b, equation (2.14) is integrable by linearization through (2.13). On the other hand, equation (2.14) violates the Painlevé property. Solving it for x'', we find a term proportional to x^2/x' (or, for that matter, to $1/x'$) which is incompatible with it.

More cases like the one above could have been derived but this is not necessary in order to prove our point. Integrability through linearization does not require the Painlevé property. On the other hand, we do not know of any systematic way to detect linearizability for a given differential system.

3. Discrete integrable systems

In the case of discrete systems, a difficulty appears from the outset in the sense that the discrete analogue of the Painlevé property is not well established. One of the properties that characterizes discrete integrable systems is that of singularity confinement [7]. While analysing a host of integrable mappings it was observed that whenever a singularity appeared at some iteration, due to the particular initial conditions, it disappeared after some further iterations. Thus, confinement would have been an excellent candidate for the role of the discrete analogue of the Painlevé property were it not for the fact that it is not sufficient. There exist mappings which have only confined singularities and which are *not* integrable [8]. Another property which has been proposed as an indicator of integrability in (rational) mappings is that of the degree growth of the iterates [15].

Let us illustrate what we mean by degree in a specific example. We consider a three-point mapping of the form $\overline{x} = f(x, \underline{x}; n)$ where f is rational in $x, \underline{x}$. (The 'bar' notation, which will be used throughout this section, is a shorthand for the up- and down-shifts in n i.e. $\overline{x} \equiv x(n+1)$, $x \equiv x(n)$, $\underline{x} \equiv x(n-1)$). Starting from some initial conditions x_0, x_1 we introduce homogeneous variables through $x_0 = p$, $x_1 = q/r$ and compute the homogeneity degree of the iterates of the mapping in q, r, to which we assign the same degree one, while p is assigned the degree zero. Other choices do exist but the result does not depend on the particular choice. While the degrees obtained do depend on it, the *growth* of the degree does not. Thus for a generic, non-integrable, mapping the degree growth of the iterates is exponential [16, 17]. In contrast, for integrable mappings, the growth is just polynomial. Moreover, a detailed analysis of discrete Painlevé equations [18] and linearizable mappings [19] has shown that the latter have even slower growth properties (which can be used not only as a detector of integrability but as an indicator of the integration method). In what follows, we shall examine the results of the application of the two methods to integrable discrete systems.

The first case we shall analyse are projective mappings [11]. In perfect analogy to the continuous case one can introduce the discrete projective Riccati equations. The starting point is a linear system for $(N+1)$ variables:

$$\overline{X}_\mu = \sum_{\nu=0}^{N} A_{\mu\nu} X_\nu \qquad \mu = 0, 1, \ldots, N. \tag{3.1}$$

Introducing again $x_\mu = X_\mu / X_0$, we obtain

$$\overline{x}_\mu = \frac{A_{\mu 0} + \sum_{\nu=1}^{N} A_{\mu\nu} x_\nu}{A_{00} + \sum_{\nu=1}^{N} A_{0\nu} x_\nu} \qquad \mu = 1, \ldots, N. \tag{3.2}$$

In fact, we have shown [12] that this system can always be rewritten as a $(N+1)$-point mapping in terms of a single object. Clearly, the case $N = 1$ is just a homographic (discrete Riccati) mapping for x_1. For $N = 2$ we finally find [20, 21]:

$$\overline{w} = \alpha + \frac{\beta}{w} + \frac{1}{w\underline{w}} \tag{3.3}$$

for a quantity w which is obtained from x_1, say, through some homography and α, β are given in terms of the $A_{\mu\nu}$. Because of the underlying linearization, any singularity appearing in the projective Riccati system is confined in one step. Moreover, the study of the degree of the iterates [19] shows that there is no growth at all: the degree is constant. Thus both criteria are satisfied in this case.

We turn now to the more interesting case of the Gambier mapping [22]. The latter is, in perfect analogy to the continuous case, a system of two (discrete) Riccati equations in cascade:

$$\overline{y} = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{3.4a}$$

$$\overline{x} = \frac{ayx + bx + cy + d}{fyx + gx + hy + k} \tag{3.4b}$$

where $\alpha, \ldots, \delta$ and $a, \ldots, k$ are all functions of the independent discrete variable n. In [22] it was shown that the system (3.4) is not confining unless the coefficients entering in the equation satisfy certain conditions. On the other hand, the same argument presented in the continuous case can be transposed here: the integration of the two Riccati equations in cascade can always be performed, through reduction to linear second-order mappings. The study of the degree growth of the iterates of (3.4) was performed in [19] where it was found that the growth is always linear, *independent* of the condition we referred to above.

This result leads naturally to the following generalization of the discrete Gambier system, the singularities of which are, in general, not confined:

$$\overline{y} = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{3.5a}$$

$$\overline{x} = \frac{a(y)x + b(y)}{c(y)x + d(y)} \tag{3.5b}$$

where $a, \ldots, d$ are polynomials in y the coefficients of which may depend on the independent variable n. The study of the degree growth of the iterates of (3.5) is straightforward. We find that the degree growth of x is linear. Again, system (3.5) can be integrated in cascade. On the other hand, equation (3.5) cannot be written as a three-point mapping for x. Indeed, if we eliminate y, $\overline{y}$ between (3.5*a*), (3.5*b*) and the upshift of the latter, we obtain an equation relating x, $\overline{x}$ and $\overline{\overline{x}}$ which is polynomial in all three variables, generically *not* linear in $\overline{\overline{x}}$. This does not define a mapping but rather a correspondence which in general leads to an exponential proliferation of the number of images and preimages. This correspondence is not integrable but this is not in contradiction with the integrability of (3.5). The two systems are not equivalent.

An $(N + 1)$-variables extension of the Gambier system can be easily produced. We have

$$\begin{aligned} \overline{x}_0 &= \frac{\alpha x_0 + \beta}{\gamma x_0 + \delta} \\ \overline{x}_\mu &= \frac{a_\mu(x_0, \ldots, x_{\mu-1})x_\mu + b_\mu(x_0, \ldots, x_{\mu-1})}{c_\mu(x_0, \ldots, x_{\mu-1})x_\mu + d_\mu(x_0, \ldots, x_{\mu-1})} \qquad \mu = 1, \ldots, N. \end{aligned} \tag{3.6}$$

Again, the degree growth of (3.6) can be computed leading to a linear growth and, once more, the singularities of (3.6) do not confine in general.

Thus, several linearizable systems can be found for which the singularity confinement gives more restricted predictions than the degree growth. We shall comment on this point in the conclusion.

A last point concerns the discrete analogues of the linearizable systems we have presented at the end of section 2. The procedure can be transposed to a discrete setting in a pretty straightforward way. We have a linear equation

$$\frac{\alpha\overline{x} + \beta + \gamma\underline{x} + \delta}{\epsilon\overline{x} + \zeta + \eta\underline{x} + \theta} = K \tag{3.7}$$

where $\alpha, \ldots, \theta$ are all functions of n with K a constant, and a nonlinear mapping

$$f(\underline{x}, x, \overline{x}; n) = M \tag{3.8}$$

where f is globally polynomial of degree two in all the x's but not more than linear separately in each of $\underline{x}$ and $\overline{x}$. Writing that the left-hand side of (3.7) is the same as that of its upshift we obtain an equation relating $\underline{x}$, x, $\overline{x}$ and $\overline{\overline{x}}$. For appropriate choices of $\alpha, \ldots, \theta$ this four-point equation can be identical (up to unimportant factors) to the four-point equation obtained from (3.8) by writing $f(\underline{x}, x, \overline{x}; n) = f(x, \overline{x}, \overline{\overline{x}}; n{+}1)$. The integration method is quite similar to that described in the continuous case. Given M, and starting with $\underline{x}$, x at some n, one obtains $\overline{x}$ from (3.8). Implementing (3.7) this fixes the value of K. From then on, one integrates the linear equation (3.7) for all n. Since the four-point equation is always satisfied, this means that f computed at any n has a constant value, which is just M, so (3.8) is satisfied.

Several mappings derived in [23] as special limits of discrete Painlevé equations can be linearized in this way. For instance, the nonlinear equation

$$\left(\frac{\overline{x} + x - a}{\overline{z}} - \frac{x}{\zeta}\right)\left(\frac{\underline{x} + x - a}{z} - \frac{x}{\zeta}\right) - \frac{x^2}{\zeta^2} = M \tag{3.9}$$

with a a constant, where z and ζ are defined from a single arbitrary function g of n through $z = \overline{g} + \underline{g}$, $\zeta = \overline{g} + g$, can be solved through the linear equation

$$\frac{A\overline{x} + B(x - a) + \overline{A}\underline{x}}{z\overline{x} + (\overline{z} + z)(x - a) + \overline{z}\underline{x}} = K \tag{3.10}$$

where $A = g^2(\overline{g} + \underline{g})$ and $B = -(\overline{g} + g)\overline{\overline{g}}\underline{g} - (\overline{\overline{g}} + \underline{g})\overline{g}g$. Mapping (3.9) is generically non-confining unless g is a constant.

4. Conclusion

In this paper we have addressed the question of integrability which does not necessitate the Painlevé property. We have found that for a large class of integrable, *linearizable* systems, the Painlevé property is not a prerequisite for integrability. The second-order Gambier system is the prototype of such a linearizable equation. Once we find that it can be linearized in the absence of the Painlevé property, it is straightforward to generalize the Gambier system and to extend it to N variables (violating the Painlevé property but preserving integrability through linearizability).

Having dispensed with the Painlevé property, it is possible to propose a new method of linearization where the derivative of a nonlinear system coincides with that of a linear one. The usefulness of this method has been illustrated through the derivation of a linearizable system which does not satisfy the Painlevé criterion.

At this point, we must stress that the Painlevé property can still be considered as a necessary condition for integrability provided we qualify the latter. The integrability with which the

Painlevé property is associated, often referred to as algebraic integrability, corresponds to the integration through IST methods. This is, for instance, the case of the transcendental Painlevé equations (or most of the integrable partial differential equations.) For these cases, the Painlevé property is necessary and we believe, sufficient. What this paper shows is that for the simpler case of linearizability, the Painlevé property is superfluous.

In the case of discrete systems the situation is more complicated. It would appear that what would play the role of the Painlevé property is singularity confinement. (The caveat is that the latter was shown not to be a sufficient condition.) Again, it turned out that singularity confinement is necessary for integrability through IST methods, such as for instance in the integration of the discrete Painlevé equations through isomonodromy techniques. However, for integrability through linearization, singularity confinement is too restrictive just like the Painlevé property. The study of the degree growth, on the other hand, shows that this criterion is more suitable for the detection of integrability in a larger sense: it identifies all linearizable systems as integrable with no restrictions whatsoever. This is at variance with the continuous case where no linearizability criterion seems to exist (at least none has been found to date). Moreover, the detailed information on the degree growth is a most useful indication of the precise integration procedure. Thus, although it is not clear whether the degree growth is the discrete equivalent to the Painlevé property it can be a most reliable integrability detector.

Acknowledgment

S Tremblay acknowledges a scholarship from Centre de Cooperation Internationale Franco-Québecois.

References

[1] Painlevé P 1902 *Acta Math.* **25** 1
[2] Ince E L 1956 *Ordinary Differential Equations* (Dover: London)
[3] Ablowitz M J and Segur H 1977 *Phys. Rev. Lett.* **38** 1103
[4] Adler M and van Moerbeke P 1988 *Algebraic Completely Integrable Systems: a Systematic Approach (Perspectives in Mathematics)* (New York: Academic)
[5] Ablowitz M J and Clarkson P A 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
[6] Kruskal M D, Ramani A and Grammaticos B 1989 *NATO ASI Series C* vol 310 (Dordrecht: Kluwer) p 321
[7] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[8] Hietarinta J and Viallet C 1998 *Phys. Rev. Lett.* **81** 325
[9] Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539
[10] Dorizzi B, Grammaticos B and Ramani A 1984 *J. Math. Phys.* **25** 481
[11] Grammaticos B, Ramani A and Winternitz P 1998 *Phys. Lett.* A 245 382
[12] Lafortune S, Grammaticos B and Ramani A 1999 *Physica* A **268** 129
[13] Gambier B 1909 *Acta Math.* **33** 1
[14] Chazy J 1911 *Acta Math.* **34** 317
[15] Bellon M P and Viallet C-M 1999 *Commun. Math. Phys.* **204** 425
[16] Arnold V I 1990 *Bol. Soc. Bras. Mat.* **21** 1
[17] Veselov A P 1992 *Commun. Math. Phys.* **145** 181
[18] Ohta Y, Tamizhmani K M, Grammaticos B and Ramani A 1999 *Phys. Lett.* A **262** 152
[19] Ramani A, Grammaticos B, Lafortune S and Ohta Y 1999 Linearisable mappings and the low-growth criterion *Preprint*
[20] Ramani A, Grammaticos B and Karra G 1992 *Physica* A **180** 115
[21] Ramani A, Grammaticos B, Tamizhmani K M and Lafortune S 1998 *Physica* A **252** 138
[22] Grammaticos B and Ramani A 1996 *Physica* A **223** 125
[23] Ramani A, Grammaticos B, Ohta Y and Grammaticos B 2000 *Nonlinearity* **13** at press